THE SIMPLEST ELEMENTARY FUNCTIONS

Compiled by Schmidt Edit, Óbuda University

POWER FUNCTIONS

$f(x) = x^{\alpha}$, $\alpha \in \mathbf{R}$ fixed

1. α is a positive even integer (e.g. $f(x) = x^2, x^4 \ldots$) **(Figure 1)**

Properties: $D = \mathbf{R}$, $R = \mathbf{R}^+ \cup \{0\} = [0, \infty)$. Bounded from below.

Strictly decreasing on $(-\infty, 0]$, strictly increasing on $[0, \infty)$. Concvex.

Having a local and global minimum and a zero at $x_0 = 0$ with the value $f(x_0) = 0$.

$\lim\limits_{\pm\infty} f(x) = +\infty$.

2. α is a positive odd integer (e.g. $f(x) = x, x^3, x^5 \ldots$) **(Figure 2)**

Properties: $D = R = \mathbf{R}$.

The graph is the identity function (an increasing line with *y* intercept 0 and slope 1) in the case of $\alpha = 1$.

Otherwise strictly increasing. Concave on $(-\infty, 0]$, convex on $[0, \infty)$.

Having an inflection and a zero at $x_0 = 0$ with the value $f(x_0) = 0$.

$\lim\limits_{-\infty} f(x) = -\infty$, $\lim\limits_{+\infty} f(x) = +\infty$

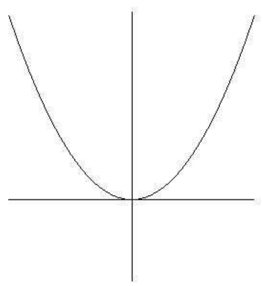

Figure 1

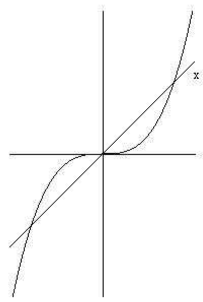

Figure 2

3. α is a negative even integer (e.g. $f(x) = \frac{1}{x^2}, \frac{1}{x^4}, \ldots$) **(Figure 3)**

Properties: $D = \mathbf{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$, $R = \mathbf{R}^+ = (0, \infty)$. Bounded from below.

Strictly increasing on $(-\infty, 0)$, strictly decreasing on $(0, \infty)$.

Convex both on $(-\infty, 0)$ and on $(0, \infty)$. Having no intersection with the axes.

$\lim\limits_{\pm\infty} f(x) = 0^+$, $\lim\limits_{0^+} f(x) = \lim\limits_{0^-} f(x) = +\infty$.

4. α is a negative odd integer (e.g. $f(x)=\frac{1}{x},\frac{1}{x^3},\ldots$) **(Figure 4)**

Properties: $D=R=\mathbf{R}-\{0\}=(-\infty,0)\cup(0,\infty)$.
Strictly decreasing both on $(-\infty,0)$ and on $(0,\infty)$.
Concave on $(-\infty,0)$, convex on $(0,\infty)$. Having no intersection with the axes.
$\lim_{-\infty} f(x)=0^-$, $\lim_{+\infty} f(x)=0^+$, $\lim_{0^-} f(x)=-\infty$, $\lim_{0^+} f(x)=+\infty$.

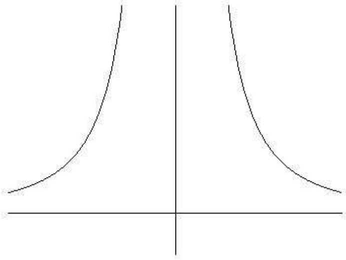

Figure 3

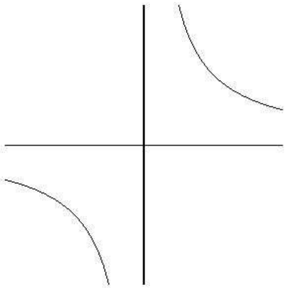

Figure 4

5. α = 0 ($f(x)=1$) **(Figure 5)**

Properties: $D=\mathbf{R}$, $R=\{1\}$. Bounded.
The graph is a line, parallel to the *x*-axis.
$\lim_{\pm\infty} f(x)=1$.

6. α is not an integer
The domain, the graph and the characterization depends on α.

For example let us see $f(x)=x^{\frac{1}{2}}=\sqrt{x}$. **(Figure 6)**
Properties: $D=R=[0,\infty)$. Bounded from below.
Strictly increasing. Concave down.
Having a global minimum and a zero at $x_0=0$ with the value $f(x_0)=0$.
$\lim_{+\infty} f(x)=+\infty$.

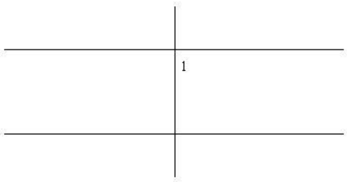

Figure 5

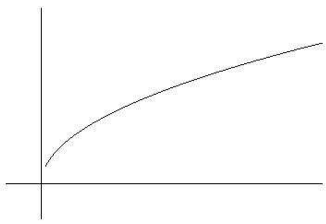

Figure 6

EXPONENTIAL FUNCTIONS

$f(x) = a^x, \quad a \in \mathbf{R}$ fixed, $a > 0, \ a \neq 1$

1. a > 1 (e.g. $f(x) = 2^x, e^x \ldots$) **(Figure 7)**

Properties: $D = \mathbf{R}, \ R = \mathbf{R}^+ = (0, \infty)$. Bounded from below.

Strictly increasing. Convex. Having no zero. Intersects the *y*-axis at $y_0 = 1$.

$\lim\limits_{-\infty} f(x) = 0^+, \ \lim\limits_{+\infty} f(x) = +\infty$.

2. 0 < a < 1 (pl. $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}, \ \left(\frac{1}{e}\right)^x = e^{-x} \ldots$) **(Figure 8)**

Properties: $D = \mathbf{R}, \ R = \mathbf{R}^+ = (0, \infty)$. Bounded from below.

Strictly decreasing. Convex. Having no zero. Intersects the *y*-axis at $y_0 = 1$.

$\lim\limits_{-\infty} f(x) = +\infty, \ \lim\limits_{+\infty} f(x) = 0^+$.

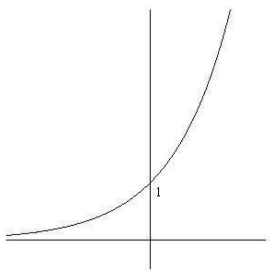

Figure 7

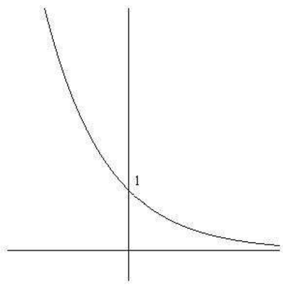

Figure 8

LOGARITHMIC FUNCTIONS

$f(x) = \log_a x, \quad a \in \mathbf{R}$ fixed, $a > 0, \ a \neq 1$

1. a > 1 (e.g. $f(x) = \log_2 x, \ \log_e x = \ln x, \ \log_{10} x = \lg x \ldots$) **(Figure 9)**

Properties: $D = \mathbf{R}^+ = (0, \infty), \ R = \mathbf{R}$.

Strictly increasing. Concave. Having a zero at $x_0 = 1$.

$\lim\limits_{0^+} f(x) = -\infty, \ \lim\limits_{+\infty} f(x) = +\infty$.

2. a < 1 (e.g. $f(x) = \log_{\frac{1}{2}} x, \ \log_{\frac{1}{e}} x \ldots$) **(Figure 10)**

Properties: $D = \mathbf{R}^+ = (0, \infty), \ R = \mathbf{R}$.

Strictly decreasing. Convex. Having a zero at $x_0 = 1$.

$\lim\limits_{0^+} f(x) = +\infty, \ \lim\limits_{+\infty} f(x) = -\infty$.

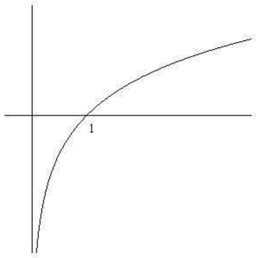

Figure 9

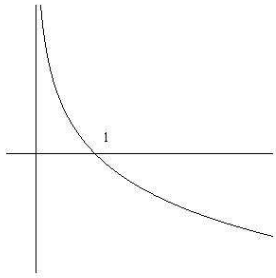

Figure 10

TRIGONOMETRIC FUNCTIONS

1. $f(x) = \sin x$ **(Figure 11)**

Properties: $D = \mathbf{R}$, $R = [-1;1]$. Bounded.

Periodic, with period 2π. Characterization on $[0, 2\pi)$:

Strictly increasing on $\left[0, \frac{\pi}{2}\right)$, strictly decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, strictly increasing on $\left[\frac{3\pi}{2}, 2\pi\right)$.

Concave on $[0, \pi)$, convex on $[\pi, 2\pi)$.

Having a local and global maximum at $x_0 = \frac{\pi}{2}$ with the value $f(x_0) = 1$. Having a local and global minimum at $x_0 = \frac{3\pi}{2}$, with the value $f(x_0) = -1$.

Having an inflection and a zero at $x_1 = 0$ and at $x_2 = \pi$.

Having no $\lim\limits_{\pm\infty} f(x)$.

2. $f(x) = \cos x$ **(Figure 12)**

Properties: $\mathrm{D} = \mathbf{R}$, $\mathrm{R} = [-1;1]$. Bounded.

Periodic, with period 2π. Characterization on $[0, 2\pi)$.

Strictly decreasing on $[0, \pi)$, Strictly increasing on $[\pi, 2\pi)$.

Concave on $\left[0, \frac{\pi}{2}\right)$, convex on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, concave on $\left[\frac{3\pi}{2}, 2\pi\right)$.

Having a local and global maximum at $x_0 = 0$,with the value $f(x_0) = 1$. Having a local and global minimum at $x_0 = \pi$, with the value $f(x_0) = -1$.

Having an inflection and a zero at $x_1 = \frac{\pi}{2}$ and at $x_2 = \frac{3\pi}{2}$.

Having no $\lim\limits_{\pm\infty} f(x)$.

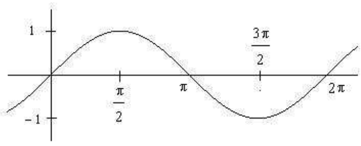

Figure 11

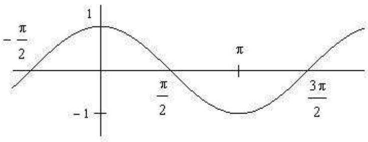

Figure 12

3. $f(x) = \tan x$ **(Figure 13)**

Properties: $D = \mathbf{R} \setminus \left\{ \frac{\pi}{2} + k \cdot \pi \right\}$ where $k \in \mathbf{Z}$, $R = \mathbf{R}$.

Periodic, with period π. Characterization on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

Strictly increasing. Concave on $\left(-\frac{\pi}{2}, 0\right]$, convex on $\left[0, \frac{\pi}{2}\right)$.

Having an inflection and a zero at $x_0 = 0$.

$\lim\limits_{-\frac{\pi}{2}^+} f(x) = -\infty$, $\lim\limits_{\frac{\pi}{2}^-} f(x) = +\infty$, but having no $\lim\limits_{\pm\infty} f(x)$.

4. $f(x) = \cot x$ **(Figure 14)**

Properties: $D = \mathbf{R} \setminus \left\{ \frac{\pi}{2} + k \cdot \pi \right\}$ where $k \in \mathbf{Z}$, $R = \mathbf{R}$.

Periodic, with period π. Characterization on $(0, \pi)$:

Strictly decreasing. Convex on $\left(0, \frac{\pi}{2}\right]$, concave on $\left[\frac{\pi}{2}, \pi\right)$.

Having an inflection and a zero at $x_0 = \frac{\pi}{2}$.

$\lim\limits_{0^+} f(x) = +\infty$, $\lim\limits_{\pi^-} f(x) = -\infty$, but having no $\lim\limits_{\pm\infty} f(x)$.

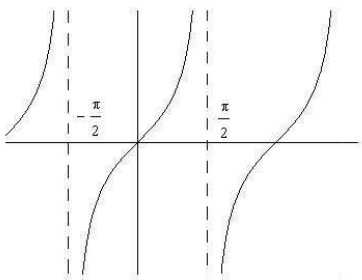

Figure 13

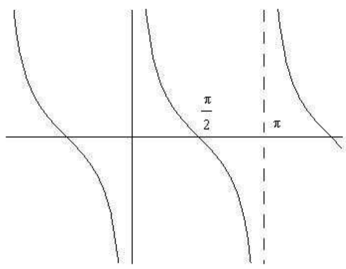

Figure 14